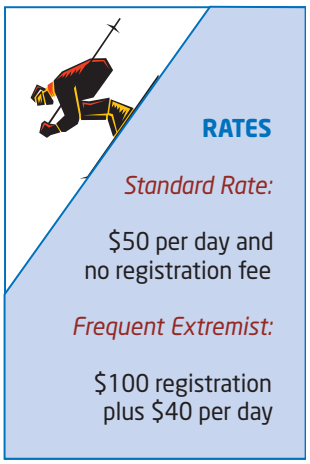
6.7 Linear Systems

Mike wants to join a ski club for the winter season. He is considering the two options shown.

Which payment option should Mike choose?



Investigate

How can you use a linear system to solve problems?

Tools

- grid paper
- ruler

Refer to the information above.

1. **a)** Write an equation that relates the total cost, C, in dollars, and the number of days, n, that Mike goes skiing if he chooses the Standard Rate.
   b) Is this a direct or a partial variation? Explain how you know.
2. Repeat step 1 for the Frequent Extremist option.
3. Graph both linear relations on the same grid. Use two different colours, one for each line. Clearly label each line with its equation. This combined graph illustrates a **linear system**.
4. **a)** Look at the point where the two lines cross. This is called the **point of intersection**. What is the value of n at this point?
   b) Look to the left of the point of intersection. Which plan is cheaper? Explain how you know.
   c) Look to the right of the point of intersection. Which plan is cheaper? Explain how you know.
   d) Find the cost of both plans at the point of intersection. What does this mean?
5. **Reflect** Is one of the two payment options clearly better than the other? Explain what additional information a skier or snowboarder needs to know to choose between the two.

linear system

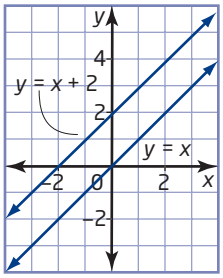
- a set of two or more linear equations that are considered simultaneously (at the same time)

point of intersection

- the point where two or more lines cross

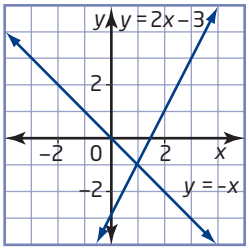
A solution to a linear system of two equations is a point or set of points that satisfy both equations. There are three different types of solutions:

- two parallel lines



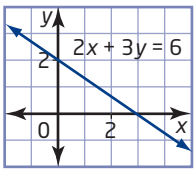
These lines never cross. There is no point that satisfies both equations. There is no solution.

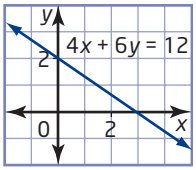
- two non-parallel lines



These lines intersect at one point: $(1, -1)$. This is the only point that satisfies both equations. There is one solution.

- two identical lines





Although the equations look different, they describe the same line. All points on one line satisfy the equation of the other. There is an infinite number of solutions.

Example 1 Solve a Linear System Graphically

a) Graph the following lines on the same grid and identify the coordinates of the point of intersection.

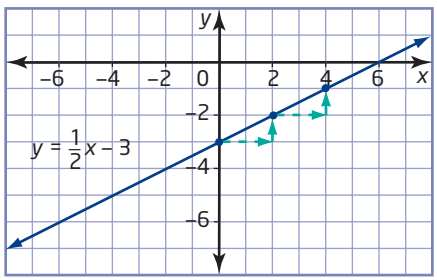
$y = \frac{1}{2}x - 3$ and $x + y = -6$

b) Verify that your solution satisfies both equations.

Solution

a) The first equation is in slope y-intercept form: $y = \frac{1}{2}x - 3$.

The slope is $\frac{1}{2}$ and the y-intercept is -3. Use this information to graph the line.



The form of the second equation makes it easy to determine its x- and y-intercepts and then use them to graph the line.

Find the x-intercept.
Substitute $y = 0$.

$$x + y = -6$$
$$x + (0) = -6$$
$$x = -6$$

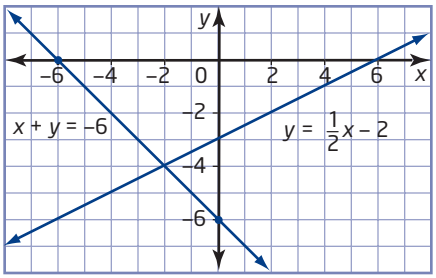
The x-intercept is -6.

Find the y-intercept
Substitute $x = 0$.

$$x + y = -6$$
$$(0) + y = -6$$
$$y = -6$$

The y-intercept is -6.

Use the intercepts to graph this line on the same grid as the first line.



The two lines intersect at the point $(-2, -4)$. This is the solution to this linear system.

b) To verify the solution, $(-2, -4)$, substitute the coordinates into both equations and check that they hold true. Use the left side/right side (L.S./R.S.) method.

Check: $y = \frac{1}{2}x - 3$

$$\text{L.S.} = y \qquad \text{R.S.} = \frac{1}{2}x - 3$$
$$= -4 \qquad = \frac{1}{2}(-2) - 3$$
$$= -1 - 3$$
$$= -4$$
$$\text{L.S.} = \text{R.S.}$$

Therefore, the point $(-2, -4)$ satisfies the equation $y = \frac{1}{2}x - 3$.

Check: $x + y = -6$

$$\text{L.S.} = x - y \qquad \text{R.S.} = -6$$
$$= (-2) + (-4)$$
$$= -6$$
$$\text{L.S.} = \text{R.S.}$$

Therefore, the point $(-2, -4)$ satisfies the equation $x + y = -6$.

The point $(-2, -4)$ satisfies both equations. $(-2, -4)$ is the correct solution to the linear system.

Example 2 Solve a Linear System Using a Graphing Calculator

A couple has budgeted $5000 for their wedding reception. Which hotel offers the better deal, and under what conditions?

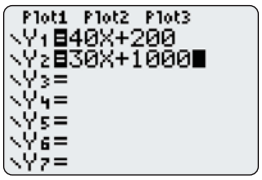
Waverly Inn	Hotel Niagara
$200 plus $40 per guest	$1000 plus $30 per guest

Solution

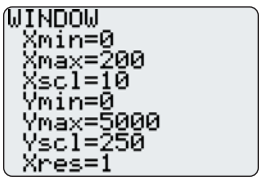
Write an equation to model each cost, C, in dollars, as it relates to the number of guests, n.
Waverly Inn: $C = 40n + 200$
Hotel Niagara: $C = 30n + 1000$

Graph the linear system using a graphing calculator:

Enter each equation using Y=.
Use x and y instead of n and C.

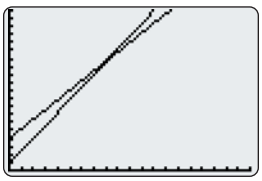


Press WINDOW and enter the settings shown.

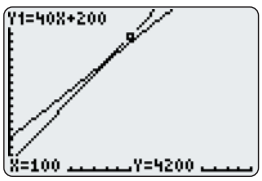


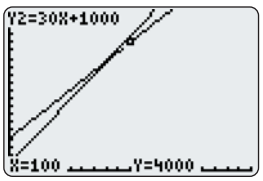
I only need to consider positive values. I'll start both axes at zero. A wedding might have about 200 guests or less, and the cost should be no greater than about $5000. I'll try these as the x- and y-scale settings.

Press GRAPH to see the linear system.



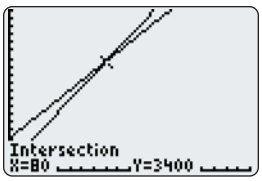
Press TRACE to read coordinates from one of the lines. Use the up and down cursor keys to switch from one line to the other.





Use the left and right cursor keys to move along the graphs.
To accurately find the intersection point,
- press 2nd [CALC]
- select **5:intersect**

The calculator will prompt you three times: for the first curve, the second curve, and a guess for the intersection point. Press ENTER three times to respond.



The two lines intersect at the point (80, 3400). At this point, Y1 and Y2 are equal. This means that if exactly 80 guests attend, the cost is the same at either hotel: $3400.

For $n < 80$ (less than 80 guests), Y1 is below Y2. This means that the cost at Waverly Inn is less than the cost at Hotel Niagara.

For $n > 80$ (more than 80 guests), the situation is reversed: Y2 is below Y1. In this case, Hotel Niagara is cheaper.

Key Concepts

- A linear system is two or more linear equations considered together.
- The solution of a linear system of two equations is the point at which their graphs intersect. The coordinates of a solution must satisfy both equations.

Communicate Your Understanding

C1 Refer to Example 2.

a) Which hotel costs less if 50 guests attend? How much less is it?

b) Which hotel costs less if 100 guests attend? How much less is it?

c) What advice would you give to a friend or relative that is considering holding a wedding at one of the two hotels?

C2 The lines $y = x - 2$ and $x + y = 6$ intersect at a point. Which is the point of intersection? Explain how you know.

A (3, 5) **B** (3, 3) **C** (2, 4) **D** (4, 2)

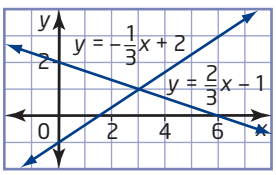
C3 Describe the steps you would use to find the solution to a linear system if you were given the two equations.

Practise

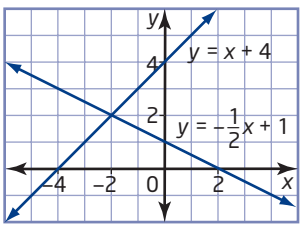
For help with questions 1 and 2, see Example 1.

1. Give the coordinates of the point of intersection of each linear system.

a)



b)



2. Solve each linear system. Verify each solution by substituting the coordinates of your solution into both equations.

a) $y = -x$ and $y = x - 6$

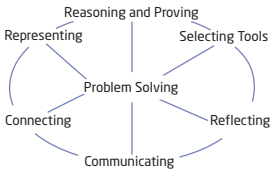
b) $x - y = 8$ and $x + 2y = 2$

c) $x + 2y = 7$ and $y = 4x - 10$

d) $y = -\frac{1}{2}x + \frac{9}{2}$ and $y = 3x - 6$

Connect and Apply

It is recommended that you use a graphing calculator or graphing software for some of these questions. See the Investigate to answer questions 3 to 5.



3. Suppose Mike went skiing six times over the winter season.

a) How much would it cost him
- under the Standard Rate option?
- under the Frequent Extremist option?

b) Which option should Mike choose in this case? Explain.

4. Suppose Mike went skiing 20 times over the winter season. Repeat question 3 for this scenario.

5. Is there a scenario in which it does not matter which option Mike chooses? If so, describe it, referring to the graph.

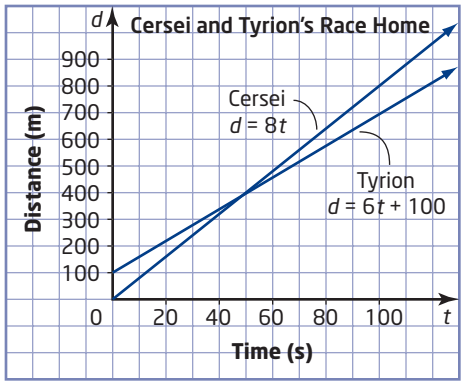
6. See Example 2. Suppose Hotel Niagara offers a special. Explain how this special may affect the couple's decision.

7. The Friendship Trail is a multi-use recreational trail that runs from Port Colborne to Fort Erie, a distance of 25 km. At 2 P.M., Debbie starts rollerblading from Port Colborne at 10 km/h. At the same time, Ken starts bicycling from Fort Erie at 20 km/h. When will they meet each other? How far from Fort Erie will they be at this time?

8. **Chapter Problem** An interesting, but unusual, feature of Jean's hometown is that two major east-west streets run parallel most of the time, but actually cross each other twice! How is this possible? Graph the following linear system and find the intersection point to discover the final two letters in the name of this city.

$x - y + 2 = 0$
$7x - 6y = 0$

9. Cersei and her brother Tyrion decide to race home. Cersei is a faster runner than Tyrion, so she gives him a head start. Their distance-time graphs are shown.



a) How much of a head start did Tyrion get?

b) How fast does Cersei run?

c) How fast does Tyrion run?

d) For what length of race will each runner win? For what length of race will they tie?

e) Explain the significance of the solution of this linear system.

10. Refer to question 9. How do your answers to part d) change if Tyrion's head start is

a) doubled?

b) cut in half?

Achievement Check

11. A recording artist is offered two deals for her fourth CD release:

- Royalty only: \$1 per CD sold
- Partial royalty: \$2000 plus \$0.50 per CD sold

a) Graph both linear relations on the same grid.

b) Find the solution to the linear system and explain what it means.

c) Sales figures for the artist's first three CDs are shown.

CD	Copies Sold
1	1500
2	3500
3	6000

Which deal do you think the artist should choose? Explain your reasoning.

Extend

12. Systems of equations can involve non-linear relations. Consider the population growth patterns of two towns since the turn of the century.

- Numberton has been growing steadily by 1000 every year, from an initial population of 25 000.
- Decimalville has been growing by 10% of its previous year's population every year, from an initial population of 15 000.

a) Copy and complete the table of values up to 15 years. Round to the nearest whole number if necessary.

Year	Numberton's Population	Decimalville's Population
0	25 000	15 000
1	26 000	16 500
2	27 000	18 150
3		

b) Graph population versus years for the towns on the same grid.

c) Classify each relation as linear or non-linear.

d) Identify the solution of this system, and explain what it means.

13. Math Contest Which is the point of intersection for the linear system $3x + 5y = 2$ and $x - 3y = 10$?

A $(4, 2)$

B $(4, -2)$

C $(-4, 2)$

D $(-4, -2)$

E $(2, -2)$

14. Math Contest Find the equation of the line that passes through the point of intersection of $-2x + 4y = 14$ and $5x - 3y = -14$, and that is perpendicular to $4x - 6y + 12 = 0$.

15. Math Contest

a) Find the point of intersection of the lines $3x + 5y = 7$ and $2x + 4y = 6$.

b) Now, find the point of intersection of $x + 5y = 9$ and $5x + 3y = 1$.

c) Investigate the point of intersection of the lines $ax + by = c$ and $dx + ey = f$, where a, b, c and d, e, f are both arithmetic sequences (an arithmetic sequence is a sequence with constant first differences). Write a summary of your findings.